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Computing realistic Reynolds-uniform error bounds for discrete derivatives of flow velocities in the boundary layer for Prandtl's problem

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SUMMARY

In this paper, we describe an experimental error analysis technique for computing realistic values of the parameter-uniform order of convergence and error constant in the maximum norm associated with a parameter-uniform numerical method for solving singularly perturbed problems. We then employ this technique to compute Reynolds-uniform error bounds in the maximum norm for appropriately scaled discrete derivatives of the numerical solutions generated by a fitted-mesh upwind finite-difference method applied to Prandtl's problem arising from laminar flow past a thin flat plate. Here the singular perturbation parameter is the reciprocal of the Reynolds number. This illustrates the efficiency of the technique for finding realistic parameter-uniform error bounds in the maximum norm for numerical approximations to scaled derivatives of solutions to problems in cases where no theoretical error analysis is available. Copyright © 2003 John Wiley & Sons, Ltd.

KEY WORDS: singular perturbation problem; experimental error analysis; Prandtl's problem; Reynolds-uniform error bounds

1. INTRODUCTION

The numerical solutions and their maximum pointwise errors for standard numerical methods applied to singularly perturbed problems depend on the singular perturbation parameter ε and the number N of mesh points. The error in the numerical solution increases as ε decreases

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to zero. Worse still, for such numerical methods, there are always values of ε for which the maximum pointwise error grows as the mesh is refined. Such behaviour of the errors is normally regarded as unacceptable in computational practice for the numerical solutions of singularly perturbed problems and especially for their derivatives. On the other hand, if the method is ε-uniform the maximum pointwise errors satisfy ε-uniform error bounds and decrease independently of ε as N grows. We emphasize that we are interested in error bounds (in the pointwise maximum norm) not only with respect to the number of nodes, but also which are valid for all values of the parameter ε. For an ε-uniform method one can formulate an error bound for the numerical solutions as follows there exist positive constants N_0, $C = C(N_0)$ and $p = p(N_0)$, all independent of N and ε, such that for all $N \geqslant N_0$.

$$\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N} \leqslant C_p N^{-p} \tag{1}$$

where u_ε is the exact solution of the continuous problem, U_ε^N is a numerical approximation on a piecewise-uniform fitted mesh Ω_ε^N (for details see, e.g. References [1, 4, 5]), $\|U_\varepsilon^N - u_\varepsilon\|_{\Omega_\varepsilon^N}$ is the maximum pointwise error of the numerical method on the mesh Ω_ε^N, C_p is the ε-uniform error constant and p is the ε-uniform order of convergence. The quantities p and C_p are called the ε-uniform error parameters. The expression in (1) can be generalized in an obvious way to discrete approximations $G(U_\varepsilon^N)$ to $g(u_\varepsilon)$, where G is a discrete form of a function g; in this paper g is always a scaled derivative.

Theoretical error analysis often does not give sharp estimates for the ε-uniform order of convergence p and almost never gives a realistic estimate of the ε-uniform error constant C_p. An underestimate of the former and an overestimate of the latter may mistakenly be taken to imply that the ε-uniform method is of no practical use. Thus, to obtain realistic ε-uniform error estimates, we are forced to adopt an experimental technique. Moreover, in many applied problems, for which the theoretical analysis of ε-uniform convergence is at present unavailable, an experimental technique may be the only option available for estimating the error. In practice, for a specific computation, the values of N and ε lie in finite ranges $R_N = \{N : \underline{N} \leqslant N \leqslant \bar{N}\}$ and $R_\varepsilon = [\underline{\varepsilon}, \bar{\varepsilon}]$, respectively, and so we want to determine estimates of p and its associated C_p such that the ε-uniform error bound $C_p N^{-p}$ in (1) is as small as possible, for all values of N and ε in the given ranges. In what follows we describe an experimental technique for computing realistic estimates of the ε-uniform error parameters p and C_p for Prandtl's problem for laminar flow past a flat plate, and we illustrate its efficiency by estimating the maximum pointwise ε-uniform error in appropriately scaled discrete derivatives of the numerical solutions. Here the Reynolds number $Re = 1/\varepsilon$ and small values of ε correspond to large values of Re. This extends to scaled discrete derivatives the analogous results in Reference [2] for the numerical solutions. Numerical approximations of derivatives are important for computing approximate values of quantities of practical interest, such as drag. Using standard numerical methods they are hard to obtain, but we show here that our robust methods do not suffer from this defect.

2. AN EXPERIMENTAL TECHNIQUE FOR COMPUTING REALISTIC PARAMETER-UNIFORM ERROR ESTIMATES

A general experimental technique for estimating the ε-uniform order of convergence p, and the ε-uniform error constant C_p, for a specific numerical method for a given singularly per-

turbed problem, is described here for cases when the numerical solutions can be computed for several values of N and ε. We assume throughout that the method is known theoretically to be ε-uniform for some, in general, unknown $p>0$ and unknown $C_p<\infty$. As is typical for techniques of this type, the arguments are heuristic. We assume that, on appropriate meshes Ω_ε^N, the piecewise linear interpolants $\bar{U}_\varepsilon^N$ of the numerical solutions U_ε^N have been determined. Then, for all integers N satisfying $N, 2N \in R_N$ and for a finite set of values $\varepsilon \in R_\varepsilon$, the maximum pointwise two-mesh differences

$$D_\varepsilon^N = \| U_\varepsilon^N - \bar{U}_\varepsilon^{2N} \|_{\Omega_\varepsilon^N} \tag{2}$$

are computed. From these values the ε-uniform maximum pointwise two-mesh differences

$$D^N = \max_{\varepsilon \in R_\varepsilon} D_\varepsilon^N \tag{3}$$

are formed for each available value of N satisfying $N, 2N \in R_N$. Approximations to the ε-uniform order of local convergence are defined, for all $N, 4N \in R_N$, by

$$p^N = \log_2 \frac{D^N}{D^{2N}} \tag{4}$$

and we take the computed ε-uniform order of convergence to be

$$p^* = \min_N p^N \tag{5}$$

Note that

$$D_\varepsilon^N = \| U_\varepsilon^N - \overline{U_\varepsilon^{2N}} \|_{\Omega_\varepsilon^N} \geqslant | \| U_\varepsilon^N - u_\varepsilon \|_{\Omega_\varepsilon^N} - \| u_\varepsilon - \overline{U_\varepsilon^{2N}} \|_{\Omega_\varepsilon^N} | \approx CN^{-p}(1-2^{-p})$$

which is used to motivate the following definitions. Corresponding to the value of p^* in (5) we calculate the quantities

$$C_{p^*}^N = \frac{D^N N^{p^*}}{1-2^{-p^*}} \tag{6}$$

and we take the computed ε-uniform error constant to be

$$C_{p^*}^* = \max_N C_{p^*}^N \tag{7}$$

The above definitions of the computed error parameters p and C_p supercede the similar definitions given in Reference [3].

The above experimental error analysis technique can be used in the following way. Suppose that we have a problem from a class of problems for which it is known theoretically that the method is ε-uniform. The technique yields values $p^*>0$ and $C_{p^*}^*$ from which we drive the realistic error bounds $C_{p^*}^* N^{-p^*}$. From these we can then determine an appropriate choice of the value of N to ensure that the method generates numerical solutions of any prerequisite guaranteed accuracy. It also enables us to generate an error table for the numerical solutions, because we can replace the unknown exact solution in the expression for the error by a numerical solution of known guaranteed accuracy generated on a sufficiently fine mesh. Such an error table gives us information about the actual, as opposed to the asymptotic, convergence behaviour of the numerical approximations to the solution of a specific problem.

On the other hand, if we are dealing with a specific problem for which it is not known if the method is ε-uniform or not, then the algorithm can yield a value $p^* > 0$ or a value $p^* \leqslant 0$. If $p^* > 0$ we conclude that, in practice, the method is ε-uniform for this problem for the particular range of ε and N used in the computations. Also we can generate an error table as before. If $p^* \leqslant 0$, then we conclude that the method is unlikely to be ε-uniform for any problem class that contains the problem. In the event that we are dealing with a problem for which we already know either the exact solution or an approximate solution with arbitrary guaranteed accuracy, then the above procedure enables us to compute an error table for the numerical solutions generated by the numerical method applied to this specific problem.

3. PRANDTL PROBLEM IN A FINITE RECTANGLE AWAY FROM THE LEADING EDGE

Consider the rectangle $\Omega = (a, A) \times (0, B)$, where a, A and B are fixed and independent of ε, a is arbitrarily small and A and B are arbitrarily large. We are interested in the following problem

$$(P_\varepsilon) \begin{cases} \text{Find } \mathbf{u}_\varepsilon = (u_\varepsilon, v_\varepsilon) \text{ such that for all } (x, y) \in \Omega \\ -\varepsilon \dfrac{\partial^2 u_\varepsilon(x, y)}{\partial y^2} + \mathbf{u}_\varepsilon \cdot \nabla u_\varepsilon(x, y) = 0 \\ \nabla \cdot \mathbf{u}_\varepsilon(x, y) = 0 \\ \mathbf{u}_\varepsilon = \mathbf{0} \text{ on } \Gamma_{\mathrm{B}} \\ u_\varepsilon = u_{\mathrm{B}} \text{ on } \Gamma_{\mathrm{L}} \cup \Gamma_{\mathrm{T}} \end{cases}$$

where u_{B} is the exact solution of the Blasius problem (see Reference [1] for details).

We now construct a direct numerical method for solving (P_ε). This is composed of a standard upwind finite-difference operator on an appropriate piecewise-uniform fitted mesh. Because the computational domain is rectangular, the piecewise-uniform fitted rectangular mesh $\Omega_\varepsilon^{\mathbf{N}}$ is the tensor product of one-dimensional meshes. That is $\bar{\Omega}_\varepsilon^{\mathbf{N}} = \bar{\Omega}_u^{N_x} \times \bar{\Omega}_\varepsilon^{N_y}$, where $\mathbf{N} = (N_x, N_y)$, $\bar{\Omega}_u^{N_x}$ is a uniform mesh with N_x mesh intervals on the interval $[a, A]$ of the x-axis, and $\bar{\Omega}_\varepsilon^{N_y}$ is a piecewise-uniform mesh with N_y mesh intervals on the interval $[0, B]$ of the y-axis, such that the subinterval $[0, \sigma]$ and the subinterval $[\sigma, B]$ are both subdivided into $\frac{1}{2} N_y$ uniform mesh intervals. We choose the transition point between the coarse and fine meshes to be

$$\sigma = \min \left\{ \frac{1}{2} B,\ \sqrt{\varepsilon} \ln N \right\}$$

Using the above piecewise-uniform fitted mesh $\Omega_\varepsilon^{\mathbf{N}}$, the problem (P_ε) is discretized by the following non-linear system of upwind finite-difference equations for the approximate velocity

$\mathbf{U}_\varepsilon = (U_\varepsilon, V_\varepsilon)$

$$(P_\varepsilon^{\mathbf{N}}) \begin{cases} \text{Find } \mathbf{U}_\varepsilon = (U_\varepsilon, V_\varepsilon) \text{ such that for all mesh points } (x_i, y_j) \in \Omega_\varepsilon^{\mathbf{N}} \\ -\varepsilon \delta_y^2 U_\varepsilon(x_i, y_j) + (\mathbf{U}_\varepsilon \cdot \mathbf{D}^-) U_\varepsilon(x_i, y_j) = 0 \\ (\mathbf{D}^- \cdot \mathbf{U}_\varepsilon)(x_i, y_j) = 0 \\ \mathbf{U}_\varepsilon = \mathbf{0} \text{ on } \Gamma_{\mathrm{B}} \\ U_\varepsilon = U_{\mathrm{B}}^{8192} \text{ on } \Gamma_{\mathrm{L}} \cup \Gamma_{\mathrm{T}} \end{cases}$$

where $\mathbf{D}^- = (D_x^-, D_y^-)$. We note that in $(P_\varepsilon^{\mathbf{N}})$ we use the known approximate boundary values U_{B}^{8192} on $\Gamma_{\mathrm{L}} \cup \Gamma_{\mathrm{T}}$ to replace the unknown exact boundary values u_{P}, where U_{B}^{8192} is the computed Blasius solution of the Prandtl problem from Reference [1]. Since $(P_\varepsilon^{\mathbf{N}})$ is a non-linear finite-difference method, it is necessary to prescribe an ε-uniform non-linear solver for computing ε-uniformly convergent approximations to its solution. The algorithm can be expressed in the following form:

$$(A_\varepsilon^{\mathbf{N}}) \begin{cases} \text{With the boundary condition } U_\varepsilon^M = U_{\mathrm{B}}^{8192} \text{ on } \Gamma_{\mathrm{L}} \\ \text{for each } i, 1 \leqslant i \leqslant N, \text{ use the initial guess } \mathbf{U}_\varepsilon^0|_{X_i} = \mathbf{U}_\varepsilon^{M_{i-1}}|_{X_{i-1}} \\ \text{and for } m = 1, \ldots, M_i \text{ solve the following} \\ \text{two point boundary value problem for } U_\varepsilon^m(x_i, y_j) \\ (-\varepsilon \delta_y^2 + \mathbf{U}_\varepsilon^{m-1} \cdot \mathbf{D}^-) U_\varepsilon^m(x_i, y_j) = 0, \quad 1 \leqslant j \leqslant N-1 \\ \text{with the boundary conditions } U_\varepsilon^m = U_{\mathrm{B}} \text{ on } \Gamma_{\mathrm{B}} \cup \Gamma_{\mathrm{T}} \\ \text{and the initial guess for } V_\varepsilon^0|_{X_1} = 0 \\ \text{Also solve the initial value problem for } V_\varepsilon^m(x_i, y_j) \\ (\mathbf{D}^- \cdot \mathbf{U}_\varepsilon^m)(x_i, y_j) = 0 \\ \text{with initial condition } V_\varepsilon^m = 0 \text{ on } \Gamma_{\mathrm{B}} \\ \text{Continue to iterate between the equations for } \mathbf{U}_\varepsilon^m \text{ until } m = M_i \\ \text{where } M_i \text{ is such that} \\ \max\left(|U_\varepsilon^{M_i} - U_\varepsilon^{M_i - 1}|_{\bar{X}_i}, \ \frac{1}{\sqrt{\varepsilon}} |V_\varepsilon^{M_i} - V_\varepsilon^{M_i - 1}|_{\bar{X}_i} \right) \leqslant \mathrm{tol} \end{cases}$$

For notational simplicity, we suppress explicit mention of the iteration superscript M_i henceforth, and we write simply $\mathbf{U}_\varepsilon$ for the solution generated by $(A_\varepsilon^{\mathbf{N}})$. We take $\mathrm{tol} = 10^{-6}$ in the computations. We note that, because there are no known theoretical results concerning the convergence of the solutions $\mathbf{U}_\varepsilon$ of $(P_\varepsilon^{\mathbf{N}})$ to the solution $\mathbf{u}_\varepsilon$ of (P_ε), we have no theoretical estimate for the pointwise error $(\mathbf{U}_\varepsilon - \mathbf{u}_\varepsilon)(x_i, y_j)$. It is for this reason that we are forced to adopt an experimental error analysis technique.

4. ESTIMATING THE PARAMETER-UNIFORM ERROR PARAMETERS

In the computations in this section we take

$$a = 0.1, \quad A = 1.1, \quad B = 1, \quad N_x = N_y = N, \quad \mathrm{tol} = 10^{-6}$$

Table I. The values of D^N, p^N, p^* and $C^N_{p^*}$ given by the experimental error analysis technique for the discrete derivatives $D_x^- U_\varepsilon$ generated from $(A_\varepsilon^{\mathbf{N}})$ applied to (P_ε) for various values N.

	Number of intervals N			
	32	64	128	256
D^N	0.763D+00	0.464D+00	0.258D+00	0.150D+00
p^N	0.72	0.85	0.78	$p^* = 0.72$
$C^N_{0.72}$	23.6	23.6	21.6	20.7

Table II. The values of D^N, p^N, p^* and $C^N_{p^*}$ given by the experimental error analysis technique for the scaled discrete derivatives $\sqrt{\varepsilon} D_y^- U_\varepsilon$ generated from $(A_\varepsilon^{\mathbf{N}})$ applied to (P_ε) for various values N.

	Number of intervals N			
	32	64	128	256
D^N	0.586D-01	0.388D-01	0.183D-01	0.107D-01
p^N	0.59	1.08	0.77	$p^* = 0.59$
$C^N_{0.3}$	1.35	1.35	0.955	0.840

The results of applying the experimental error analysis technique to the appropriately scaled discrete derivatives of the numerical solutions generated by the algorithm $(A_\varepsilon^{\mathbf{N}})$ applied to problem (P_ε) are summarized in Tables I and II for $R_N = \{32, 64, 128, 256, 512\}$ and $R_\varepsilon = \{2^{-r} : r = 0(1)20\}$. We remark that the derivative of u_ε in the direction orthogonal to Γ_B (i.e. $\partial u_\varepsilon / \partial y$) tends to infinity as $\varepsilon \to 0$, at a rate proportional to $1/\sqrt{\varepsilon}$. Hence, it is necessary to consider approximations to the scaled derivatives $\sqrt{\varepsilon} \partial u_\varepsilon / \partial y$, which are bounded independently of ε.

From the results in these tables we deduce the following Reynolds-uniform maximum error parameters and pointwise error bounds for all $M \geqslant 8 \ln N$ and all $N \geqslant 32$

$$\left\| D_x^- U_\varepsilon - \frac{\partial u_\varepsilon}{\partial x} \right\|_{\bar{\Omega}} \leqslant 23.6 N^{-0.72}$$

$$\sqrt{\varepsilon} \left\| D_y^- U_\varepsilon - \frac{\partial u_\varepsilon}{\partial y} \right\|_{\bar{\Omega}} \leqslant 1.35 N^{-0.59}$$

In the next two tables we compute maximum pointwise error bounds for specific choices of N and we compare them with the computed maximum pointwise errors found by replacing the unknown exact solution in the expression for the error by a semi-analytic solution on a fine mesh computed by Blasius' method (see Reference [1] for details).

We see from Tables III and IV that in each case the computed Reynolds-uniform maximum pointwise error bound is greater than the corresponding computed error, and so it is an upper bound. Moreover, in Table III we see that this upper bound is never more than 1.6 times the error, while in Table IV the corresponding multiple is 2.5. Thus, for each discrete derivative the upper bound may be regarded as realistic. Since $\partial v_\varepsilon / \partial y = -\partial u_\varepsilon / \partial x$ the results in Table III

Table III. Computed Reynolds-uniform maximum pointwise error bound $C^*_{p^*}N^{-p^*}$ for $\sup_\varepsilon \|D_x^- U_\varepsilon - \partial u_\varepsilon/\partial x\|$ generated from $(A_\varepsilon^{\mathbf{N}})$ applied to (P_ε) compared to the computed error from a Blasius solution of known guaranteed accuracy for various values of N.

	Number of intervals N				
	32	64	128	256	512
$23.6N^{-0.72}$	1.95	1.18	0.72	0.44	0.26
$\sup_\varepsilon \|D_x^- U_\varepsilon - \partial_x U_{\mathrm{B}}^{8192}\|$	1.21	0.80	0.50	0.30	0.18
Ratio	1.61	1.48	1.44	1.45	1.47

Table IV. Computed Reynolds-uniform maximum pointwise error bound $C^*_{p^*}N^{-p^*}$ for $\sup_\varepsilon \sqrt{\varepsilon}\|D_y^- U_\varepsilon - \partial u_\varepsilon/\partial y\|$ generated from $(A_\varepsilon^{\mathbf{N}})$ applied to (P_ε) compared to the computed error from a Blasius solution of known guaranteed accuracy for various values of N.

	Number of intervals N				
	32	64	128	256	512
$1.35N^{-0.59}$	0.175	0.116	0.077	0.051	0.034
$\sup_\varepsilon \sqrt{\varepsilon}\|D_y^- U_\varepsilon - \partial_y U_{\mathrm{B}}^{8192}\|$	0.118	0.073	0.043	0.025	0.014
Ratio	1.48	1.59	1.79	2.04	2.43

apply to this derivative. Finally, the derivative $\partial v_\varepsilon/\partial x$ presents special problems, which are not yet fully resolved; see Reference [1] for more details.

We emphasize that we are endeavouring to produce an upper bound on the errors for all $\varepsilon > 0$ and all $N \geqslant N_0$. Since this range of N (for N_0 not too large) may include regions where the scheme has not yet attained asymptotic behaviour, we expect that these upper bounds will be far from the actual errors in certain cases. One aim of generating these error bounds is to enable their use to produce guarantees for an extensive range of N and ε on the final error in the computed quantities, although these guarantees may be unduly pessimistic in particular cases (i.e. for particular choices of N and ε). Another aim is to permit an estimate of the number of nodes N required to guarantee that a level of accuracy desired by the user of the method will be obtained.

It is worth noting that, strictly speaking, the experimental error analysis technique is not known to be applicable to the numerical solutions generated by the algorithm $(A_\varepsilon^{\mathbf{N}})$ applied to problem (P_ε), because currently there is no theoretical error analysis known to be available. Nevertheless it is clear from the results presented here that the experimental error analysis technique provides a means to estimate the accuracy of the numerical approximations to the derivatives of the unknown continuous solution even in cases when no theoretical error estimates are known.

5. CONCLUSION

We described an experimental error analysis technique for computing realistic values of the parameter-uniform order of convergence and error constant in the maximum norm associated

with a parameter-uniform numerical method for solving singularly perturbed problems. We then employed this technique to compute Reynolds-uniform error bounds in the maximum norm for appropriately scaled discrete derivatives of the numerical solutions generated by a fitted-mesh upwind finite-difference method applied to Prandtl's problem arising from laminar flow past a thin flat plate. Here the singular perturbation parameter is the reciprocal of the Reynolds number. This illustrated the efficiency of the technique for finding realistic parameter-uniform error bounds in the maximum norm for numerical approximations to scaled derivatives of solutions to problems in cases where no theoretical error analysis is available.

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